Newton's Shell Theorem – Rigorous Proof

Statement. For a spherically symmetric thin shell of radius R and total mass M: (1) any point outside the shell (distance r > R) experiences gravity as if all mass were concentrated at the center; (2) any point strictly inside the shell (r < R) experiences zero net gravitational force from the shell. We present three complementary proofs: (A) via Gauss's law for gravity, (B) via direct integration of the gravitational potential, and (C) Newton's geometric cancellation argument.

A. Proof via Gauss's Law for Gravity

Consider the gravitational field *g*(\mathbf{r}) generated by the shell. Gauss's law for gravity states ■_S g · dA = -4πG M_enc for any closed surface S, where M_enc is the mass enclosed by S. By spherical symmetry, for a sphere of radius r centered at the shell's center, the field is radial and uniform on S: ■ g · dA = g(r) · 4π r^2. • If r > R, the sphere encloses the entire shell: M_enc = M, so g(r) = -GM / r^2 (radially inward). • If r < R, the sphere encloses no mass: M_enc = 0, hence g(r) = 0. Therefore the field outside is identical to that of a point mass at the center and vanishes inside, proving both claims.

B. Direct Integration of the Gravitational Potential

Let the shell have uniform surface mass density σ = M / (4πR^2). Place the field point P at distance r from the center O. Using spherical coordinates with polar angle θ measured from the OP axis, the ring at polar angle θ has area dA = 2π R^2 sinθ dθ, mass dm = σ dA, and each point on the ring is at distance d(θ) = sqrt(R^2 + r^2 - 2 R r cosθ) from P. The potential contribution is dΦ = -G dm / d(θ). Integrating over the shell gives
Φ(r) = -G σ ∫_{0}^{π} (2π R^2 sinθ dθ) / sqrt(R^2 + r^2 - 2 R r cosθ).

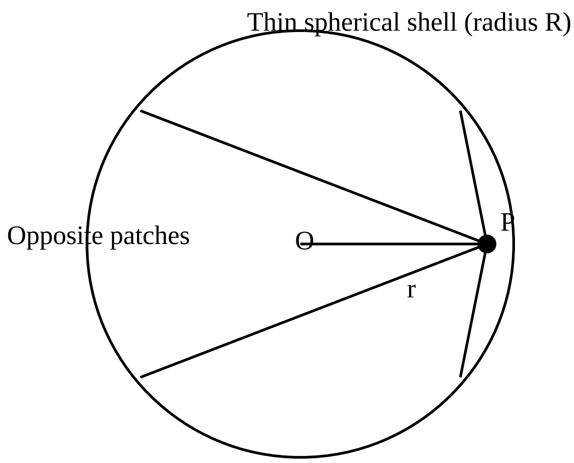
A standard approach uses the spherical-harmonic expansion of the Newtonian kernel: for r > R, 1 / d(θ) = ∑_{■=0}^∞ (R^■ / r^{■+1}) P_■(cosθ); for r < R, 1 / d(θ) = ∑_{■=0}^∞ (r^■ / R^{■+1}) P_■(cosθ). The ring integral averages P_■(cosθ) over the sphere; all terms with ■ ≥ 1 vanish by orthogonality, leaving only the ■ = 0 term (P_0 = 1). Thus:
Φ(r) = -G σ · (2π R^2) · ∫_{0}^{π} sinθ dθ × { ext{■=0 term only}} = ■egin{cases} -GM / r, & r > R, \ -GM / R, & r < R. \end{cases}

Because g(r) = -dΦ/dr, the exterior field is g(r) = -GM / r^2 and the interior field vanishes, since the potential is constant (-GM/R) inside.

C. Newton's Geometric Cancellation (Field Inside a Shell)

Pick any point P inside the shell (r < R). Consider a narrow cone of solid angle dΩ centered on the line OP. The cone intersects the shell in two small patches on opposite sides of the shell along OP. Their areas scale like dA ∝ R^2 dΩ / cosα, while their distances to P differ: one is near (distance d■), the other far (distance d■). Because the shell is thin, the masses in the two patches are proportional to their areas; the gravitational forces scale as dm / d^2. Geometry of similar triangles gives dm■ / d■^2 = dm■ / d■^2 and the forces are directed in opposite radial directions, hence they cancel exactly. Summing over all cones (i.e., solid angles) yields zero net force at P.

Figure: Geometry for cancellation in a thin spherical shell.



Conclusion. All three arguments rely on the inverse-square law and spherical symmetry. They establish that a uniform thin spherical shell produces a gravitational potential equal to that of a point mass at the center outside the shell, and a constant potential inside; the resulting field is `g(r) = -GM/r^2` for $r \geq R$ and zero for $r < R$. Extensions: the theorem holds for any spherically symmetric mass distribution when considering the field outside (it depends only on enclosed mass), and for concentric shells the interior field depends only on mass inside the radius of interest.